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# SELECTED RECENT DEVELOPMENTS IN CHANCE-CONSTRAINED PROGRAMMING

by

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13. ABSTRACT

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## ABSTRACT

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## 1. The Origins of Chance-Constrained Programming

The fundamentals of chance-constrained programming (CCP) are now nearly two decades old. During this time, theoretical research and practical applications have clarified and strengthened the original ideas, injected new insights into the meaning of chance constraints, and led to the development of important new classes of probabilistic programming models. Regretably, this development has been accompanied by a proliferation of terminology, symbology and essentially equivalent model formulations. Consequently few attempts have been made at presenting a unified summary of CCP results. In the remainder of this paper we present a summary of the most significant recent results. First, however, we recall the circumstances surrounding CCP's original development.

CCP was originated by Charnes, Cooper and Symonds [59] in the context of developing a planning and operations model for Standard Oil of New Jersey for the scheduling of heating oil manufacture, storage and distribution with weather-dependent demand. Initially the situation was modeled as a constrained game against nature [46]. The constrained game approach, however, was not entirely satisfactory. While the constraints on the permissible strategies of nature were substantially better than none at all, the "clairvoyant malevolence" of nature implicit in the minimax solution did not seem appropriate to the behaviour of weather.

Another important consideration was that while the policy of the company as the major supplier of a product of national importance was to supply all emergent demand, one could find random considerations of weather whose associated demand would be impossible for the company to meet from its available resources. Indeed, it was discovered on further investigation that several incidents had actually occurred in the past in which the company could not meet the demand. On these occasions, extra-ordinary measures, outside the company's control, had to be taken to alleviate the crises.

A model was therefore sought which (a) did not place nature in the role of a clairvoyant malevolent competitor, (b) would express a "policy" as a rule which was to hold almost all, but not all, the time, and (c) would develop a plan and an operationally implementable procedure for "most", but not all randomly emergent situations. These challenges chance-constrained programming met through (a) representation of nature more classically via joint probability distributions, (b) representation of a "policy" as a "chance constraint," and (c) posing the problem of chance-constrained programming as determining from a preassigned admissible class a vector of stochastic decision rules which would satisfy the chance constraints and optimize the expected value of a preassigned functional.

It is clear that within these prescriptions chance-constrained programming may be considered a method of tremendous flexibility with innumer-

able individual possible variations or models. A chance constraint, e.g., the requirement that a given function of the sought vector satisfy a given condition to at least a prescribed level of probability, admits as many interpretations as does the probability operator, e.g., total or conditional, etc. Again, where a system of constraints is desired to hold, chance constraints may be imposed individually on the members of the system ("individual chance constaints") or jointly ("joint chance constraints"). One might employ functionals ranging from minimization of expected costs to maximization of the probability of some event. Additionally, the preassigned class of admissible stochastic decision rules, which represent the operational prescriptions of the model, can range from preset parameters to be determined ("zero-order decision rules") to the most general class of "informationally feasible" measurable functions of sample points of the random variables, and, non-operationally, to "clairvoyant" rules. In any particular situation to be modeled, of course, the choice is in the hands of the statistician to select the blend of representation of the real features of the situation with an effectively operable and analyzable structure.

From the work of Charnes, Cooper and Symonds it was immediately apparent that the CCP approach held potentially great applicability to a far wider class of problems than those of scheduling the production of heating oil. Thus, theoretical work continued in parallel with the original application effort in order to establish the beginnings of a framework for future theoretical

developments as well as additional applications. (see [2] for an example of early results of such work). Since these beginnings, interest in CCP has mushroomed and the list of individuals making contributions grows longer annually.

To illustrate more concretely the nature of the opportunities and the difficulties of chance-constrained programming, we turn now to a mathematical representation of the problem of CCP.

## 2. Definition of Chance-Constrained Programming

The CCP literature contains a wide variety of differing formulations of varying degrees of generality. All these formulations, however, possess several common ingredients, including decision rules which must be selected from some prescribed class, an extremization objective function, linear or nonlinear constraints of the conventional deterministic type, and chance constraints which stipulate that certain relations (equations or inequalities) between various random variables and the decision rules must be satisfied with at least a specified level of probability.

In order to state the CCP problem in a typical but general form we reproduce here the CCP problem as first presented in [10]. Virtually every formulation in the literature can be obtained by considering various special cases of this problem.

We use $b = (b_1, \ldots, b_m)$ to denote a vector of random variables $b_i$, $i = 1 \ldots m$ with a known joint distribution function,

$$F(t) = F(t_1, \ldots, t_m) = P\{ b_i \leq t_i,\ i = 1, \ldots, m\}.$$

From its definition F is right continuous. The subset of m space which is the range of the random vector b will be denoted $\Omega$. The operators P and E will denote respectively probability and expectation integrals; these are to be evaluated with respect to the Lebesgue-Stieltjes measure (also denoted by F) induced by the function F. The letter $\alpha$ will denote throughout a given, fixed real number such that $0 < \alpha \leq 1$.

For any univariate distribution function G, the $\alpha$ fractile of G is given by

$$G^{-1}(\alpha) = \inf\{s: G(s) \geq \alpha\}.$$

Thus, in general, $G^{-1}(G(t)) \leq t$ and $G(G^{-1}(\alpha)) \geq \alpha$, but if G is continuous at $G^{-1}(\alpha)$ then $G(G^{-1}(\alpha)) = \alpha$.

A decision rule (or rule) x(t), where $X: R^m \to R^n$, is a vector function which is a member of some prescribed nonempty subclass C of the Lebesgue measurable functions. Examples of the class C which we will consider are:

(a) $C_0 = \{x: x(t) = x_0 \text{ for all } t\}$, the class of constant or zero-order rules.

(b) $C_L = \{x: x(t) = Dt + d$, where D is m x n, d is m x 1$\}$, the class of linear or <u>first order</u> rules.

(c) $C_M$ or $C_{PA}$ or $C_{PL}$, the class of measurable or piecewise analytic or piecewise linear rules, respectively.

(d) $C_k = \{x: x_i(t)$, the i-th component of x, is an arbitrary measurable function of $t_1, \ldots, t_{j_i}$, some $j_i$, and a constant function of $t_j$ for $j > j_i\}$, the class of k-stage or k-order rules.

Note that if $j_i = 0$ for all i then $C_k = C_0$, while if $j_i = m$ for all i then $C_k = C_m$.

For our purposes, the interpretation of a decision rule is that x(b) represents a decision which will be taken as a function of the random variable in b. In [33, 34] the class of clairvoyant decision rules, in which the decision rule is permitted to be a function of all the random variables involved in the model, was introduced. This class of rules is important for in principle evaluation of the importance (or value) of information and in obtaining bounds on the optimal value of the objective function. The chance-constrained programming problem with clairvoyant decision rules contains as a special case the stochastic linear programming models of Tintner [55, 56]. The special case results from the fact that the chance constrained formulation also admits constraints which are required to hold only with probability at least $\alpha < 1$, whereas the constraints of stochastic linear programming are required to hold with probability one.

The extreme opposite class of decision rules, in terms of the amount of information which is employed by the decision maker in the decision rule, is the class $C_0$ of "zero-order" or "pre-assigned parameter" rules. The term "zero-order", like the term "first-order", refers to what sort of

mathematical functions of the random variables are involved in the rules. In informational terms, the class of rules $x(t) = x_0$ for all t might better be called the class of "non-adaptive" rules. Decision rules in classes intermediate between the "non-adaptive" and "clairvoyant" classes permit adaptation in response to observed values of the random variables.

When information (observed sample values of the random variable ) becomes available stage-by-stage, $C_k$, the class of k stage rules, may correspond in informational terms to the class of <u>informationally feasible</u> rules. In [17, 18, 19] such rules have been considered in the context of n period models in which the decision rule for the $k^{th}$ period is permitted to be a variable function of the random variables of periods 1 to k-1, but must be a constant function of the random variables of the $k^{th}$ and future periods. Thus, the $k^{th}$ period decision is taken in response to <u>all</u> information available then e.g. the observed values of the random variables of periods 1 to k-1 and in anticipation of values which may arise for the random variables in periods k through n. One may consider the informationally feasible rules, and all other n period rules in which less information is used, to be non-randomized rules in the sense that we require the decision x(b) to be determined once the appropriate random variables have been observed. Randomized rules can of course be comprehended under this rubric by the device that the randomization at each stage can be considered as realized by random <u>drawing</u> or sampling of other random variables prior to the onset of this stage.

Within the above framework we define the chance-constrained programming problem as follows:

Select, if one exists, a decision rule $x(b)$ with $x \in C$ which yields an infimum to

$$Ef(x(b), b)$$

subject to

$$P\{g_j(x(b), b) \geq 0\} \geq \alpha_j, \quad j = 1, \ldots, J \tag{1}$$

and

$$x(b) \in K \text{ with probability one (w.p.1).}$$

In (1), the functions $f$, $g_j$ are assumed to be Borel measurable and to be such that $f\colon R^n \times R^m \to R^1$ and $g_j\colon R^n \times R^m \to R^{m_j}$. The set $K$ is assumed to be some nonempty subset of $R^n$.

Example 1: Take $C = C_k$ (with $k = 2$) and $j_i = 0$ for $i = 1, \ldots, n_1$ and $j_i = m$ for $i = n + 1, \ldots, n$. Then

$$x(t) = [x_1, \ldots, x_{n_1}, y_1(t), \ldots, y_{n-n_1}(t)].$$

Let

$$K = \{(x, y(t)) : Dx \geq d,\ x \geq 0,\ y(t) \geq 0\}$$

and set

$$f(x(b), b) = c_1 x + c_2 y(b)$$

for fixed vectors $c_1$ and $c_2$.

Let $J = 2$ and let $g_1 = Ax + My(b) - b = -g_2$. Let $\alpha_1 = \alpha_2 = 1$.

Then (1) becomes the two-stage linear programming under uncertainty problem:

$$\min c_1x + Ec_2y(b)$$

$$Dx \geq d, \qquad x \geq 0 \qquad (2)$$

subject to

$$Ax + My(b) = b, \quad y(b) \geq 0 \text{ w.p. } 1.$$

Example 2: Use the same choice of C as in Example 1, and take $b = (b^1, b^2)$ where $b^1 = (b_1, \ldots, b_m)$, $b^2 = (b_{m+1}, \ldots, b_{\bar{m}})$ so that $x(b) = (x, y(b^1))$. Choose $K = \{(x,y): x \geq 0\}$. Choose $J = \bar{m} + (n - n_1)$. Let $m_j = 1$ for all $j = 1, \ldots, J$. Let $g_j = A^j_{11}X - b_j$, $j = 1, \ldots, m$; $g_j = A^j_{21}X + A^j_{22}Y(b^1) - b_j$, $j = m+1, \ldots, \bar{m}$; $g_j = Y_{j-\bar{m}}(b^1)$, $j = \bar{m}+1, \ldots, \bar{m} + (n - n_1)$; where the superscript $j$ denotes the $j$-th row of the matrix.

Let $f = C_1X + C_2Y(b^1)$ for fixed vectors $C_1$ and $C_2$. Problem (1) then becomes

$$\min C_1X + EC_2Y(b^1)$$

subject to

$$P(A^j_{11}X \geq b_j) \geq \alpha_j, \qquad j = i, \ldots, m,$$

$$P(A^j_{21}X + A^j_{22}Y(b^1) \geq \alpha_j, \qquad j = m+1, \ldots, \bar{m}, \quad (3)$$

and

$$P(Y^{(b^1)}_{j-\bar{m}} \geq 0) \geq \alpha_j, \qquad j = \bar{m}+1, \ldots, n.$$

This is the two-stage E model of chance-constrained programming.

If we interpret the P operation in the constraints of (3) to be a total

probability operator, then we get the chance constrained model with total probability constraints [see 8,19, 22]. If we interpret P to mean that we compute the probability using the conditional distribution of $b^2$ given $b^1$ then we get the chance constrained model with conditional probability constraints [see 17, 33, 34].

Example 3: If we use the same functions as in Example 2, but redefine f to be equal to 1 if $c_1x + c_2y(b^1) \geq L_0$ and 0 otherwise, for some prescribed $L_0$, we obtain the P-model objective of chance-constrained programming namely,

$$\min P\{C_1X + C_2Y(b^1) \geq L_0\}.$$

P models have been discussed in [8, 35].

Example 4: If we now define f to be the function $[C_2Y(b^1) - E(C_2Y(b^1))]^2$, then the objective becomes that of the V model of chance-constrained programming [6], namely,

$$\min \operatorname{var}[C_1X + C_2Y(b^1)].$$

Example 5: If we consider the case of (1) in which J=1, but in which $m_1$ is not necessarily equal to one, the chance constraint becomes

$$P\{g_i(x(b), b) \geq 0, \ i = 1, \ldots, m_1\} \geq \alpha,$$

where

$$g_i: R^m \times R^n \to R^1.$$

Thus we obtain a joint chance constraint. Models with joint chance constraints have been considered in several recent papers, notably [11, 13, 42, 43].

For use in the following sections we write here some nonlinear and linear joint chance-constrained problems with decision rule class $C_M$:

$$\min Ef(x(b))$$

$$x(b) \in K \text{ w.p. } 1$$

subject to

$$P\{g_i(x(b)) \geq b_i, \text{ all } i = 1, \ldots, m_1\} \geq \alpha \tag{4a}$$

and

$$x \in C_M.$$

$$\min Ecx(b)$$

$$x(b) \in K \text{ w.p. } 1$$

subject to

$$P\{Ax(b) \geq b\} \geq \alpha \tag{4b}$$

and

$$x \in C_M.$$

Note that (4a) is specialized to the extent that $g_i(x(b), b)$ is written (with slight abuse of notation) as $g_i(x(b)) - b_i$ and $f = f(x(b))$, while in (4b), $g_i(x(b), b)$ is $a^i x(b) - b_i$, where $a^i$ is the i-th row of the matrix A.

## 3. Properties of Optimal Decision Rules

In recent years, considerable effort has been devoted to establishing properties of optimal decision rules for various probabilistic programming

models, but particularly for the two stage and n-stage models described in Examples 1 and 2 above. Results on the two stage linear programming under uncertainty problem (Example 1) can be found in [49, 51, 52, 53, 54]. These results show that the two stage problem can be rewritten in the form of an equivalent deterministic convex nonlinear programming problem. Several such deterministic equivalents have been established and algorithms for their solution have been, or are being, developed.

A major issue raised by the two stage problem involves determining conditions under which the second stage constraints

$$My(b) = b - Ax$$
$$y(b) \geq 0$$

are consistent for all possible values of the random vector b and all first stage decisions x. In the early work on the two stage problem it was simply assumed that the second stage constraints were consistent. Subsequently Kall [5] established the following result:

Kall's Theorem: For $Ax = b$ to have a non-negative solution $x$ for all $b \in R^m$ it is necessary and sufficient that there exist $\mu_j \geq 0$ and $\lambda_j < 0$ such that

$$\sum_{j=m+1}^{m+k} \mu_j A_j = \sum_{j=1}^{m} \lambda_j A_j$$

where $A_j$ is the j-th column of $(m + k) \times n$ matrix A and $A_1, \ldots, A_m$ are linearly independent.

A stronger result, which we have presented for some years in our classes but not previously published in the literature, is the following:

Theorem

A necessary and sufficient condition that $Ax = b$, $x \geq 0$ have solutions for in linearly independent vectors $b^{(1)}, \ldots, b^{(m)}$, and their negatives is that there exist non-negative $nx(m + k)$ matrices $N^+$, $N^-$ such that $AN^+ = I$, $AN^- = -I$.

Corollary

Under the hypothesis of the above theorem, non-negative solution vectors x exist for all vectors $b \in R^m$, to wit,

$$x = N^+ b^+ + N^- b^-$$

where

$$b^{\pm} = 1/2 \{b \pm |b|\}.$$

Solution theorems and characterizations of the optimal decision rules for the n-stage chance-constrained programming problem are given in [17, 18, 19]. For the two stage problem (Example 2) with the class of informationally feasible rules, the main result is the following:

Theorem

There exists an optimal vector of second stage decisions $y(b^1)$ which is a piecewise linear function of the vector of conditional fractile points $F_2^{-1}(1 - \alpha_2)$ and the vector of first stage decisions x.

More useful results for special forms of $A_{22}$ and specific distributions of the random vector $b^1$ can be found in the literature. However, an algorithm for finding the optimal piecewise linear rule does not exist for the general two stage problem.

## 4. Acceptance Regions in Chance Constrained Programming

In section 1 we noted the highly desirable aspects of chance-constrained programming in allowing one to formulate plans and procedures for "most", but not all, of the time, where the precise exceptional situations are locked up implicitly in the problem and need not be known a priori. The fact, however, that an optimal class of stochastic decision rules is determined on the viable system sample points (and indeed rigorous specification of the problem requires explicit consideration of the unviable or "rejected" regions) suggest analogies with the Neyman-Pearson theory of testing statistical hypotheses. Thus the notion of "acceptance regions" for chance constraints has recently been formulated by Charnes, Kirby, and Raike [10] and applied to derive essential simplifications in chance-constrained programming theory, particularly in the more recondite areas of joint and conditional chance constraints.

To introduce the concept of an acceptance region we proceed as follows:

Let $g(x, b): R^n \times R^m \to R^{m_1}$. Let $\overline{x}(t): R^m \to R^n$ be some prescribed decision rule which is a member of the class C.

Definition: The acceptance region for the chance constraint $P\{g(x(b), b) \geq 0\} \geq \alpha$ under the decision rule $\overline{x}$ is defined as

$$A_g(\overline{x}) = \{t \in R^m: g(\overline{x}(t), t) \geq 0\}.$$

<u>Definition</u>: Let $V_g(\bar{x}) = \{t \in R^m : g(\bar{x}(t), t) < 0\}$. $V_g(\bar{x})$ will be termed the rejection region for the chance constraint $P\{g(x(b), b) \geq 0\} \geq \alpha$ under the decision rule $\bar{x}$. The subscript g and the argument $\bar{x}$ will occasionally be suppressed if no confusion would result.

We summarize some pertinent and elementary observations about the rejection and acceptance regions in the form of a lemma. The proof is obvious and will be omitted.

<u>Lemma</u>

(a) For any $x \in C$, $V_g(x)$ and $A_g(x)$ are measurable sets.

(b) $A_g(x) = V_g^c(x)$, the complement of $V_g(x)$.

(c) $x \in C$ is feasible (i.e., $x(b) \in K$ w.p. 1 and $P\{g(x(b), b) \geq 0\} \geq \alpha$ if, and only if, $P\{b \in V_g(x)\} \leq 1 - \alpha$, or equivalently $P\{b \in A_g(x)\} \geq \alpha$, $x(b) \in K$ w.p. 1.

The regions $V_g(x)$ and $A_g(x)$ are thus simply the sets of those points for which a given decision rule x violates or satisfies, respectively, the relation $g \geq 0$.

Before proceeding to establish properties of optimal acceptance and rejection regions for (1), we make the following two assumptions about the functions f, and $g_j$, $j = 1, \ldots, J$ and the set K.

Assumption: The functions $-f(x,t)$ and $g_j(x,t)$ are assumed to be concave and twice continuously differentiable functions x for each t and to be continuous as functions of t. The set K is assumed to be defined as $K = \{x: h_i(x) \geq 0, \; i \in I\}$, where the $h_i$ are twice continuously differentiable concave functions. The set $\{x: h_i(x) \geq 0, \; g_j(x,t) \geq 0$ for all i and j$\}$ is assumed to have a nonempty interior for each t. The set

$$\{x: g_j(x,t) \geq 0, \; j = 1, \ldots, J\} \cap K \cap \{x: f(x,t) \leq u\}$$

is assumed to be compact for all constants u.

Assumption: It is assumed that one of the following holds.

(i) For each t, at least one of the f, or $-h_i$, are strictly convex functions.

(ii) Nonnegativity constraints are present for all variables.

(iii) There are n-independent linear constraints present; Note that (ii) is a special case of (iii).

Then, if (1) is "regularized", the following result can be established [10]:

Theorem

Let $A = \{A_j, \; j = 1, \ldots, J\}$ be any collection of measurable sets. Then there exists a measurable rule $x_A{}^*$ for (1) such that $x_A{}^*(b) \in K$ w.p. 1 and $A_{g_j}(x_A{}^*) \supset A_j$ for each j, and such that $f(x_A{}^*(b), b) \leq f(x(b), b)$ a.e. for all measurable x having $A_{g_j}(x) \supset A_j$ for each j and $x(b) \in K$ w.p. 1.

As a direct result of this theorem we have the following:

Corollary

If there exists a rule $x \in C_M$ which is feasible and optimal in the class $C_M$ for (1), then for some $A = \{A_1, \ldots, A_J\}$ the rule $x_A{}^*(t)$ is also feasible and optimal for (1).

The importance of this corollary is that it says that if an optimal rule exists for (1), then $x_{A^*}(t)$ given by the above Theorem is also an optimal rule for (1) for some set of acceptance regions $A_j$, $j = 1, \ldots, J$. This means we can view (1) as a problem of finding an optimal set of acceptance regions rather than an optimal rule. Thus we have the following important result:

Theorem

Suppose that there exists an optimal rule for (1) with $C = C_M$. Then under A1-A3, (1) may be replaced by the equivalent problem.

Find an $A = \{A_1, \ldots, A_J\}$ with each $A_j$ measurable which yields

$$\min_A Ef(x_A{}^*(b), b)$$

subject to (5)

$$P\{b \in A_j\} \geq \alpha_j, \qquad j = 1, \ldots, J.$$

Any A which yields a minimum in (5) will be termed a collection of optimal acceptance regions. Thus, we may speak freely and interchangeably about either optimal $C_M$ rules for (1) or optimal (acceptance) regions in (5).

In [10, 42, 43] it is shown how the latter may be used to advantage in characterizing optimal classes of decision rules for certain problems, and in constructing explicit solutions to others. These results are particularly useful in problems which have only a single chance constraint or in which all the random variables involved in the problem are discrete.

## 5. Chance-Constrained Programming and Games with Random Payoff Matrices

We define a two person, zero sum random payoff game to be a two person game in which the elements $a_{ij}$ of the mxn payoff matrix A are random variables. We assume that the distribution function G of the matrix A is known; the distribution of each $a_{ij}$ is the appropriate marginal distribution.

If player I plays row i and player II plays column j, the payoff from player II to player I is determined by observing the random variable $a_{ij}$. In particular the actual payoff on any play of the game will be $a_{ij}(w)$ where w is selected from the domain of $a_{ij}$ according to the known probability distribution of $a_{ij}$.

Let $X = (x_1, x_2, \ldots, x_m)$, where $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, be a mixed strategy for player I. Let $Y = (y_1, y_2, \ldots, y_n)$, where $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, be a mixed strategy for player II. Let Z(X, Y) be the observed payoff when player I and player II use mixed strategies X and Y respectively. Let u, v be independent random variables such that $P(u=1) = x_i$ $i = 1, 2, \ldots m$

and $P(v=j) = y_j$ $j = 1, 2, \ldots, n$; and u and v are independent of the random variables $a_{ij}$ for every i and j. Player I plays according to the sample value of the random variable u and player II according to that of v with each player unaware of the sample his opponent draws. Then if u=i and v=j, the payoff is given by the random variable $a_{ij}$. It is then straightforward to verify that Z(X, Y) has the following distribution

$$P(Z(X,Y) < t) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(a_{ij} < t)\, x_i y_j$$

We assume that both players know the distribution function G of the payoff matrix A.

One criterion for optimization is as follows:

Player I assigns a weight $w(\beta)$ to each $\beta$ in the range of payoff levels $[\beta_1, \beta_2]$ such that $\int_{\beta_1}^{\beta_2} w(t)\,dt = 1$. The weight $w(\beta)$ indicates the importance player I attaches to each payoff $\beta$ in the interval $[\beta_1, \beta_2]$. Thus player I chooses the mixed strategy, $X^*$, which maximizes the weighted average of the probability of his obtaining a payoff level at least $\beta \in [\beta_1, \beta_2]$ no matter what strategy player II uses. Mathematically, the problem player I wants to solve is:

$$\max_X \min_Y \int_{\beta_1}^{\beta_2} P(Z(X,Y) \geq t)\, w(t)dt = \min_Y \int_{\beta_1}^{\beta_2} P(Z(X^*,Y) \geq t)\, w(t)dt.$$

It can be shown that this problem reduces to the following linear program:

$$\begin{array}{lll} \max_X \alpha & & \\ \text{such that} \quad \sum_{i=1}^{m} x_i \left[\int_{\beta_1}^{\beta_2} P(a_{ij} \geq t)\, w(t)\,dt\right] \geq \alpha & & \text{for every } j \\ \sum_{i=1}^{m} x_i = 1, \qquad x_i \geq 0 & & \text{for every } i \end{array} \tag{6}$$

In the special case where we define w(t) to be the Diracdelta function with mass point at $\beta$, problem (6) reduces to that of finding the optimal strategy for player I in the deterministic game with payoff matrix $\{P(a_{ij} \geq \beta\}$.

Suppose now that rather than using the optimality criterion of (6) player I wants to maximize his payoff level $\beta$, subject to the constraint that he achieve the payoff level $\beta$ with at least a prescribed probability $\alpha$, no matter what strategy his opponent uses. We suppose $\alpha > 0$ since otherwise the problem is unbounded. This problem can be expressed mathematically as:

$$\max \beta$$

$$\text{such that} \qquad \min_{Y} (P(Z(X,Y) \geq \beta) \geq \alpha \tag{7}$$

which is equivalent to:

$$\max \beta$$

$$\text{such that} \qquad \sum_{i=1}^{m} x_i P(a_{ij} \geq \beta) \geq \alpha \qquad \text{for every } j \tag{8}$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0 \qquad \text{for every } i$$

where $\alpha$ is given constant $0 < \alpha \leq 1$.

Because of the form of the constraints of (8) the problem is not only nonlinear in the variables $x_i$ ($i=1, 2, \ldots, m$) and $\beta$ but also the region of feasible solutions may not form a convex set. Despite the problems of nonconvexity, an algorithm for the solution of (8) is given in [58].

In papers now in preparation, various results are developed for models in which:

(a) The players have no information about the distribution function $F(A)$ of the random matrix A.

(b) The players have a partial ordering on the probabilities of the states of nature of A.

(c) The players know bounds on the probabilities of the various states of nature. That is, the players know that $a_k \leq p(k) \leq b_k$ where $a_k$, $b_k$ are given, $k = 1, 2, \ldots, K$.

(d) The players have complete information about the distribution $F(A)$.

In addition, models are being developed which involve other optimality criteria and in which additional constraints are placed on the possible strategies of the players. These latter models are extensions to random payoff games of the original work of Charnes [56] on constrained deterministic zero sum games.

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